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The Asymptotic Behaviors of
Triple Hypergeometric Series, F_G , F_K , F_N and F_R
Near Boundaries of Their Convergence Regions

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1. Introduction

In the series of papers [3] - [6], [8] - [15], the author and his co-authors have established various asymptotic behaviors of hypergeometric series near boundary points of convergence regions. (see also [7]) The present paper is devoted to produce more six properties {(FG1), (FG2), (FK), (FN1), (FN2) and (FR), below} for the Lauricella series of three variables, F_G , F_K , F_N and F_R , which are defined as follows [16]:

(1.1) $F_G(\alpha,\alpha,\alpha,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+n+p}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma_1)_m(\gamma_2)_{n+p}m!n!p!} x^m y^n z^p,$$

in $|x| + |y| < 1, \ |x| + |z| < 1,$

(1.2) $F_K(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p m!n!p!} x^m y^n z^p,$$

in $|x| < 1,\ |y| < 1,\ |z| < (1-|x|)(1-|y|)$,

(1.3) $F_N(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_n(\alpha_3)_p(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p} m!n!p!} x^m y^n z^p,$$

in $|x| + |z| < 1,\ |y| < 1$,

(1.4) $F_R(\alpha_1,\alpha_2,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+p}(\alpha_2)_n(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p} m!n!p!} x^m y^n z^p,$$

in $\sqrt{|x|} + \sqrt{|z|} < 1,\ |y| < 1$,

where $(\alpha)_m$ means the usual Pochhammer symbol defined by $\Gamma(\alpha+m)/\Gamma(\alpha)$.

In the following establishment we shall make use of the following two formulas for the Gauss series ${}_2F_1$ [4] and the second Appell series F_2 [10]:

(1.5) $${}_2F_1(a,b;a+b;1-\rho) = -\Gamma\left[\begin{matrix}a+b\\ a,b\end{matrix}\right] [2\gamma + \psi(a) + \psi(b) + \log\rho] + o(1),$$

$(\rho \to +0)$,

$$(1.6)\quad F_2(a,b_1,b_2;c_1,c_2;x-\rho,1-x-\delta)$$

$$= \Gamma\begin{bmatrix} c_1, c_1-a-b_1 \\ c_1-a, c_1-b_1 \end{bmatrix} x^{-a}\ {}_3F_2\begin{bmatrix} a, a-c_1+1, a+b_1-c_1; \frac{x-1}{x} \\ a+b_1-c_1+1, c_2; \end{bmatrix}$$

$$+ \Gamma\begin{bmatrix} c_1, c_2 \\ a, b_1-1, b_2+1 \end{bmatrix} \frac{c_1-b_1}{c_1-a-b_1+1}\, x^{b_1-c_1+1}\, (1-x)^{c_1-a-b_1+1}$$

$$\cdot\ {}_4F_3\begin{bmatrix} c_1-b_1+1, 2-b_1, 1, 1; \frac{x-1}{x} \\ c_1-a-b_1+2, b_2+1, 2; \end{bmatrix}$$

$$- \Gamma\begin{bmatrix} c_1, c_2 \\ a, b_1, b_2 \end{bmatrix} x^{b_1-c_1}\, (1-x)^{c_1-a-b_1}$$

$$\cdot\ [2\gamma + \psi(a+b_1-c_1) + \psi(b_2) - \log(1-x) + \log(\rho+\delta)]$$

$$+ o(1), \quad (\rho \to +0,\ \delta \to +0)$$

with $a+b_1+b_2 = c_1+c_2$, $b_2 \neq$ integer, $c_1-a-b_1 \neq$ integer and $1/2 < x < 1$, where the generalized hypergeometric series ${}_pF_q$ of single variable and the second Appell series F_2 of two variables are defined by

$$(1.7)\quad {}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z) = {}_pF_q\begin{bmatrix} \alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q; \end{bmatrix} z = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_m}{\prod_{j=1}^{q} (\beta_j)_m} \frac{z^m}{m!},$$

$$(1.8)\quad F_2(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_n m!n!} x^m y^n, \quad |x| + |y| < 1,$$

ψ is the psi function and γ is the Euler-Mascheroni constant. When $q = p-1$, (1.7) converges in $|z| < 1$ (cf. [16]). Here and in what follows, we write, for brevity,

$$\Gamma\left[\begin{matrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_q\end{matrix}\right] = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(\beta_1)\cdots\Gamma(\beta_q)} .$$

The formula (1.6) holds valid also for $0 < x < 1/2$ by exchanging b_1 with b_2 and c_1 with c_2.

2. The Behaviors of the Series F_G

Let us first consider the series F_G defined in (1.1) near the points on the plane $x + z = 1$ $(x,z > 0)$ and on the edge $x + y = 1$, $x + z = 1$ $(x,y,z > 0)$ of the boundary of its convergence region. The results are:

(FG1) $F_G(a,a,a,b_1,b_2,b_3;c_1,c_2,c_2;x,y,1-x-\rho)$

$$= \Gamma\left[\begin{matrix}c_2,c_2-a-b_3\\ c_2-a,c_2-b_3\end{matrix}\right](1-x)^{-a}\, F^{1:2;1}_{0:2;1}\left[\begin{matrix}a:a-c_2+1,a+b_3-c_2;\quad b_2;\\ -:\ a+b_3-c_2+1,c_1;c_2-b_3;\end{matrix}\,\frac{x}{x-1},\frac{y}{1-x}\right]$$

$$+ \Gamma\left[\begin{matrix}c_1,c_2\\ a,b_3-1,b_1+1\end{matrix}\right]\frac{c_2-b_3}{c_2-a-b_3+1}\, x^{c_2-a-b_3+1}(1-x)^{b_3-c_2+1}$$

$$\cdot F^{1:3;1}_{0:3;1}\left[\begin{matrix}c_2-b_3+1:\quad 2-b_3,1,1;\quad b_2;\\ \text{——}\ :c_2-a-b_3+2,b_1+1,2;c_2-b_3;\end{matrix}\,\frac{x}{x-1},\frac{y}{1-x}\right]$$

$$- \Gamma\begin{bmatrix} c_1, c_2 \\ a, b_3, b_1 \end{bmatrix} x^{c_2-a-b_3}(1-x)^{b_2+b_3-c_2}(1-x-y)^{-b_2}$$

$$\cdot\ [2\gamma + \psi(a+b_3-c_2) + \psi(b_1) - \log x + \log \rho]$$

$$+ o(1), \quad (\rho \to +0)$$

with $a+b_1+b_3 = c_1+c_2$, $b_1 \neq$ integer, $a+b_3-c_2 \neq$ integer and $0 < x < 1/2$, $x < y < 1$.

Note. The formula (FG1) holds valid also for $1/2 < x < 1$ by exchanging b_1 with b_3 and c_1 with c_2.

(FG2) $F_G(a,a,a,b_1,b_2,b_3;c_1,c_2,c_2;x,1-x-\rho,1-x-\rho)$

$$= \Gamma\begin{bmatrix} c_1, c_1-a-b_1 \\ c_1-a, c_1-b_1 \end{bmatrix} x^{-a}\ {}_3F_2\begin{bmatrix} a, a-c_1+1, a+b_1-c_1; \dfrac{x-1}{x} \\ a+b_1-c_1+1, c_2; \end{bmatrix}$$

$$+ \Gamma\begin{bmatrix} c_1, c_2 \\ a, b_1-1, b_2+b_3+1 \end{bmatrix} \frac{c_1-b_1}{c_1-a-b_1+1} x^{b_1-c_1+1}(1-x)^{c_1-a-b_1+1}$$

$$\cdot\ {}_4F_3\begin{bmatrix} c_1-b_1+1, 2-b_1, 1, 1; \dfrac{x-1}{x} \\ c_1-a-b_1+2, b_2+b_3+1, 2; \end{bmatrix}$$

$$- \Gamma\begin{bmatrix} c_1, c_2 \\ a, b_1, b_2+b_3 \end{bmatrix} x^{b_1-c_1}(1-x)^{c_1-a-b_1}$$

$$\cdot\ [2\gamma + \psi(a+b_1-c_1) + \psi(b_2+b_3) - \log(1-x) + \log\rho]$$

$$+\ o(1), \quad (\rho \to +0)$$

with $a+b_1+b_2+b_3 = c_1+c_2$, $b_2+b_3 \neq$ integer, $a+b_1-c_1 \neq$ integer and $1/2 < x < 1$.

Note. The formula (FG2) holds valid also for $0 < x < 1/2$ by exchanging b_1 with b_2+b_3 and c_1 with c_2.

Proof of (FG1). Expanding F_G in the series of y, we have its representation in terms with F_2:

(2.1) $F_G(\alpha,\alpha,\alpha,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,z)$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta_2)_m}{(\gamma_2)_m m!} F_2(\alpha+m,\beta_1,\beta_3;\gamma_1,\gamma_2+m;x,z)y^m$$

Then the relations (2.1) and (1.6) lead us to for $a+b_1+b_3 = c_1+c_2$

(2.2) $F_G(a,a,a,b_1,b_2,b_3;c_1,c_2,c_2;x,y,1-x-\rho)$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b_2)_m}{(c_2)_m m!} F_2(a+m,b_1,b_3;c_1,c_2+m;x,1-x-\rho)y^m$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b_2)_m}{(c_2)_m m!}\, y^m \left\{ \Gamma\left[\begin{matrix} c_2+m, c_2-a-b_3 \\ c_2-a, c_2-b_3+m \end{matrix}\right] (1-x)^{-a-m}\right.$$

$$\cdot\ {}_3F_2\left[\begin{matrix} a+m, a-c_2+1, a+b_3-c_2; \\ a+b_3-c_2+1, c_1; \end{matrix} \frac{x}{x-1}\right]$$

$$+\ \Gamma\left[\begin{matrix} c_2+m, c_1 \\ a+m, b_3-1, b_1+1 \end{matrix}\right] \frac{c_2-b_3+m}{c_2-a-b_3+1}\, x^{c_2-a-b_3+1}(1-x)^{b_3-c_2-m+1}$$

$$\cdot\ {}_4F_3\left[\begin{matrix} c_2-b_3+1+m, 2-b_3, 1, 1; \\ c_2-a-b_3+2, b_1+1, 2; \end{matrix} \frac{x}{x-1}\right]$$

$$-\ \Gamma\left[\begin{matrix} c_2+m, c_1 \\ a+m, b_3, b_1 \end{matrix}\right] x^{c_2-a-b_3}(1-x)^{b_3-c_2-m}$$

$$\left.\cdot\ [2\gamma + \psi(a+b_3-c_2) + \psi(b_1) - \log x + \log \rho]\right\}$$

$$+\ o(1)$$

which deduces the formula (FG1) in terms of the Kampé de Fériet series (cf. [1], [16]):

$$(2.3)\quad F^{p:r;u}_{q:s;v}\left[\begin{matrix} \alpha_1,\cdots,\alpha_p:\gamma_1,\cdots,\gamma_r;\eta_1,\cdots,\eta_u; \\ \beta_1,\cdots,\beta_q:\delta_1,\cdots,\delta_s;\zeta_1,\cdots,\zeta_v; \end{matrix} x, y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_{m+n} \prod_{j=1}^{r} (\gamma_j)_m \prod_{j=1}^{u} (\eta_j)_n}{\prod_{j=1}^{q} (\beta_j)_{m+n} \prod_{j=1}^{s} (\delta_j)_m \prod_{j=1}^{v} (\zeta_j)_n} \frac{x^m y^n}{m!n!}\ .$$

Here, the Kampé de Fériet series $F^{1:2;1}_{0:2;1}\left[x,y\right]$ and $F^{1:3;1}_{0:3;1}\left[x,y\right]$ in (FG1) converge in $|x| + |y| < 1$.

Proof of (FG2). By virtue of the series expansion of F_G in terms of the Appell series F_1:

(2.4) $$F_G(\alpha,\alpha,\alpha,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta_1)_m}{(\gamma_1)_m m!} F_1(\alpha+m,\beta_2,\beta_3;\gamma_2;y,z)x^m$$

and by the relation [1]

(2.5) $$F_1(\alpha,\beta_1,\beta_2;\gamma;x,x) = {}_2F_1(\alpha,\beta_1+\beta_2;\gamma;x),$$

we have the formula

(2.6) $$F_G(\alpha,\alpha,\alpha,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,y) = F_2(\alpha,\beta_1,\beta_2+\beta_3;\gamma_1,\gamma_2;x,y),$$

where the Appell series F_1 is defined by [1]

(2.7) $$F_1(\alpha,\beta_1,\beta_2;\gamma;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma)_m\, m!n!} x^m y^n.$$

Thus the relation (FG2) follows immediately from (1.6), (2.7) and [1]

(2.8) $$F_2(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y) = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta_1)_m}{(\gamma_1)_m m!}\, {}_2F_1(\alpha+m,\beta_2;\gamma_2;y)x^m.$$

3. The Behavior of the Series F_K

Now let us treat the series F_K near the surface $z = (1-x)(1-y)$ $(x,y,z > 0)$, then we have:

(FK) $F_K(a_1,a_2,a_2,b_1,b_2,b_1;c_1,b_2,c_3;x,y,(1-x)(1-y)-\rho)$

$$= \Gamma\left[\begin{matrix} c_1, c_1-a_1-b_1 \\ c_1-a_1, c_1-b_1 \end{matrix}\right] x^{-b_1} {}_3F_2\left[\begin{matrix} b_1, b_1-c_1+1, a_1+b_1-c_1; \frac{x-1}{x} \\ a_1+b_1-c_1+1, c_3; \end{matrix}\right]$$

$$+ \Gamma\left[\begin{matrix} c_1, c_3 \\ b_1, a_1-1, a_2+1 \end{matrix}\right] \frac{c_1-a_1}{c_1-a_1-b_1+1} x^{a_1-c_1+1} (1-x)^{c_1-a_1+b_1+1}$$

$${}_4F_3\left[\begin{matrix} c_1-a_1+1, 2-a_1, 1, 1; \frac{x-1}{x} \\ c_1-a_1-b_1+2, a_2+1, 2; \end{matrix}\right]$$

$$- \Gamma\left[\begin{matrix} c_1, c_3 \\ a_1, a_2, b_1 \end{matrix}\right] x^{a_1-c_1} (1-x)^{c_1-a_1-b_1}$$

$$\cdot \left[2\gamma + \psi(a_1+b_1-c_1) + \psi(a_2) - \log(1-x) - \log(1-y) + \log\rho\right]$$

$$+ o(1), \quad (\rho \to +0)$$

with $a_1+a_2+b_1 = c_1+c_3$, $a_2 \neq$ integer, $a_1+b_1-c_1 \neq$ integer and $1/2 < x < 1$, $0 < y < 1$.

Note. The formula (FG2) holds valid also for $0 < x < 1/2$ by exchanging a_1 with a_2 and c_1 with c_3.

Proof. The series F_K may be expanded by the Gauss series as

(3.1) $F_K(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_n(\beta_1)_{m+n}}{(\gamma_1)_m(\gamma_3)_n m!n!} \, {}_2F_1(\alpha_2+m,\beta_2;\gamma_2;y)\, x^m z^n.$$

Then by noting the relation [2]

(3.2) $${}_2F_1(\alpha,\beta;\beta;z) = (1-z)^{-\alpha},$$

we have the formula

(3.3) $F_K(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\beta_2,\gamma_3;x,y,z)$

$$= (1-y)^{-\alpha_2}\, F_2(\beta_1;\alpha_1,\alpha_2;\gamma_1,\gamma_3;x,\frac{z}{1-y}),$$

which together with the relation (1.6) leads us to (FK).

4. The Behaviors of the Series F_N

Following two formulas for F_N defined in (1.3) will be derived:

(FN1) $F_N(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2;x,y,1-x-\rho)$

$$= \Gamma\begin{bmatrix} c_1,c_1-a_1-b_1 \\ c_1-a_1,c_1-b_1 \end{bmatrix} x^{-b_1}\, {}_3F_2\begin{bmatrix} b_1,b_1-c_1+1,a_1+b_1-c_1; & \frac{x-1}{x} \\ a_1+b_1-c_1+1,c_2; & \end{bmatrix}$$

$$+ \Gamma\left[\begin{matrix} c_1, c_2 \\ b_1, a_1-1, a_3+1 \end{matrix}\right] \frac{c_1-a_1}{c_1-a_1-b_1+1} x^{a_1-c_1+1} (1-x)^{c_1-a_1-b_1+1}$$

$$\cdot\ {}_4F_3\left[\begin{matrix} c_1-a_1+1, 2-a_1, 1, 1; \\ c_1-a_1-b_1+2, a_3+1, 2; \end{matrix} \frac{x-1}{x}\right]$$

$$+ \frac{a_2 b_2}{c_2}\, y\, F^{(3)}\left[\begin{matrix} -::-; & - ;b_1:a_1;a_2+1,b_2+1,1;a_3; \\ -::-;c_2+1; & -:c_1; \qquad\qquad 2;\ -; \end{matrix} x, y, 1-x\right]$$

$$- \Gamma\left[\begin{matrix} c_1, c_2 \\ a_1, a_3, b_1 \end{matrix}\right] x^{a_1-c_1} (1-x)^{c_1-a_1-b_1}$$

$$\cdot\ [2\gamma + \psi(a_1+b_1-c_1) + \psi(a_3) - \log(1-x) + \log\rho]$$

$$+ o(1), \quad (\rho \to +0)$$

with $a_1+a_3+b_1 = c_1+c_2$, $a_3 \neq$ integer, $a_1+b_1-c_1 \neq$ integer and $1/2 < x < 1$, $0 < y < 1$.

Note. The formula (FN1) holds valid also for $0 < x < 1/2$ by exchanging a_1 with a_3 and c_1 with c_2.

(FN2) $F_N(a_1, a_2, a_3, b_1, b_2, b_1; c_1, a_2+b_2, a_2+b_2; x, 1-\rho, z)$

$$= \Gamma\left[\begin{matrix} a_2+b_2 \\ a_2, b_2 \end{matrix}\right] \frac{a_3 b_1}{a_2 b_2}\, z\, F^{1:1;3}_{0:1;3}\left[\begin{matrix} b_1+1:a_1; & a_3+1,1,1; \\ \text{—}\ :c_1; & a_2+1,b_2+1,2; \end{matrix} x, z\right]$$

$$- \Gamma\left[\begin{matrix} a_2+b_2 \\ a_2, b_2 \end{matrix}\right] {}_2F_1(a_1, b_1; c_1; x) \ [2\gamma + \psi(a_2) + \psi(b_2) + \log \rho]$$

$$+ o(1), \quad (\rho \to +0).$$

Proof of (FN1). By the series representation of F_N:

(4.1) $$F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z)$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_2)_m (\beta_2)_m}{(\gamma_2)_m \, m!} F_2(\beta_1, \alpha_1, \alpha_3; \gamma_1, \gamma_2+m; x, z) y^m$$

$$= F_2(\beta_1; \alpha_1, \alpha_3; \gamma_1, \gamma_2; x, z)$$

$$+ \frac{\alpha_2 \beta_2}{\gamma_2} y \sum_{m=0}^{\infty} \frac{(\alpha_2+1)_m (\beta_2+1)_m}{(\gamma_2+1)_m (2)_m} F_2(\beta_1, \alpha_1, \alpha_3; \gamma_1, \gamma_2+1+m; x, z) y^m$$

and the relation (1.6), the formula (FN1) follows, where the Srivastava series $F^{(3)}$ is defined by [16]:

(4.2) $$F^{(3)}\left[\begin{matrix} (\alpha) :: (\beta^1); (\beta^2); (\beta^3) : (\gamma^1); (\gamma^2); (\gamma^3); \\ (\delta) :: (\eta^1); (\eta^2); (\eta^3) : (\zeta^1); (\zeta^2); (\zeta^3); \end{matrix} \; x, y, z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \Lambda(m, n, p) \frac{x^m y^n z^p}{m! n! p!},$$

where

$$\Lambda(m,n,p) = \frac{\prod_{j=1}^{A} (\alpha_j)_{m+n+p} \prod_{j=1}^{B_1} (\beta_j^1)_{m+n} \prod_{j=1}^{B_2} (\beta_j^2)_{n+p} \prod_{j=1}^{B_3} (\beta_j^3)_{p+m}}{\prod_{j=1}^{D} (\delta_j)_{m+n+p} \prod_{j=1}^{E_1} (\eta_j^1)_{m+n} \prod_{j=1}^{E_2} (\eta_j^2)_{n+p} \prod_{j=1}^{E_3} (\eta_j^3)_{p+m}}$$

$$\cdot \frac{\prod_{j=1}^{C_1} (\gamma_j^1)_m \prod_{j=1}^{C_2} (\gamma_j^2)_n \prod_{j=1}^{C_3} (\gamma_j^3)_p}{\prod_{j=1}^{F_1} (\zeta_j^1)_m \prod_{j=1}^{F_2} (\zeta_j^2)_n \prod_{j=1}^{F_3} (\zeta_j^3)_p},$$

and (α) abbreviates the set of A parameters $\alpha_1, \cdots, \alpha_A$ and similar interpretations for (β_1), (β_2) etc. Here, the Srivastava series $F^{(3)}$ in (FN1) converges in $|x| + |z| < 1$, $|y| < 1$.

Proof of (FN2). The series expansion

(4.3) $$F_N(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_m(\alpha_3)_n(\beta_1)_{m+n}}{(\gamma_1)_m(\gamma_2)_n m!n!} \; {}_2F_1(\alpha_2,\beta_2;\gamma_2+n;y)\, x^m z^n$$

and (1.5) imply the formula (FN2), where the Kampé de Fériet series $F^{1:1;3}_{0:1;3}\left[x,y \right]$ converges in $|x| + |y| < 1$.

5. The Behavior of the Series F_R

For the series F_R we have:

(FR) $F_R(a_1,a_2,a_1,b_1,b_2,b_1;c_1,a_2+b_2,a_2+b_2;x,1-\rho,z)$

$$= \Gamma\begin{bmatrix} a_2+b_2 \\ a_2,b_2 \end{bmatrix} \frac{a_1b_1}{a_2b_2}\, z\, F^{2:0;2}_{0:1;3}\begin{bmatrix} a_1+1,b_1+1: -; \quad 1,1; \\ \text{———} \quad :c_1;a_2+1,b_2+1,2; \end{bmatrix} x,z$$

$$- \Gamma\begin{bmatrix} a_2+b_2 \\ a_2,b_2 \end{bmatrix} {}_2F_1(a_1,b_1;c_1;x)\ [2\gamma + \psi(a_2) + \psi(b_2) + \log\rho]$$

$$+ o(1), \quad (\rho \to +0).$$

Proof. By noting the series representation of F_R:

(5.1) $F_R(\alpha_1,\alpha_2,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_{m+n}(\beta_1)_{m+n}}{(\gamma_1)_m(\gamma_2)_n m!n!}\ {}_2F_1(\alpha_2,\beta_2;\gamma_2+n;y)\,x^m z^n$$

we have

$F_R(a_1,a_2,a_1,b_1,b_2,b_1;c_1,a_2+b_2,a_2+b_2;x,1-\rho,z)$

$$= {}_2F_1(a_2,b_2;a_2+b_2;1-\rho) \sum_{m=0}^{\infty} \frac{(a_1)_m(b_1)_m}{(c_1)_m m!}\, x^m$$

$$+ \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n+1}(b_1)_{m+n+1}}{(c_1)_m(a_2+b_2)_{n+1}m!(n+1)!}\ {}_2F_1(a_2,b_2;a_2+b_2+n+1;1-\rho)x^m z^{n+1},$$

which implies (FR) by virtue of (1.5) and the Gauss formula [2]

$$(5.2) \quad {}_2F_1(\alpha,\beta;\gamma;1) = \Gamma\begin{bmatrix} \gamma,\gamma-\alpha-\beta \\ \gamma-\alpha,\gamma-\beta \end{bmatrix} \quad \text{for} \quad \mathrm{Re}(\gamma-\alpha-\beta) > 0.$$

Here, the Kampé de Fériet series $F^{2:0;2}_{0:1;3}\left[x,y\right]$ in (FR) converges in $\sqrt{|x|} + \sqrt{|y|} < 1$.

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